Method of Variation of Parameters

Consider a second order linear differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x) \quad \text{--- (1)}$$

When the general solution of the reduced equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \quad \text{--- (2)}$$

(i,e; complementary function of (1)) is known.

Let the general solution of the eq^n (2) is

$$y = c_1 y_1(x) + c_2 y_2(x) \quad \text{--- (3)}$$

Where c_1 and c_2 are constants, y_1 and y_2 are two linearly independent solution of (2).

We shall replace the constants c_1 and c_2 by unknown functions $v_1(x)$ and $v_2(x)$.

$$\therefore\ y_p = v_1 y_1 + v_2 y_2 \quad \text{--- (4)}$$

Satisfies (1). This will be a particular solution of (1)

Differentiate (4), we obtain.

$$y_p' = (v_1 y_1' + v_2 y_2') + \underbrace{(v_1' y_1 + v_2' y_2)}_{0} \quad \text{--- (5)}$$

We choose v_1 and v_2 in such a manner that

$$v_1' y_1 + v_2' y_2 = 0 \quad \text{--- (6)}$$

Again differentiating (5), we have

$$y_p'' = (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2') \quad \text{--- (7)}$$

$\because$ y_p is a solution of (1), Therefor

$$\frac{d^2 y_p}{dx^2} + P(x) \frac{dy_p}{dx} + Q(x) y_p = R(x)$$

$$\Rightarrow (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2')$$

$$+ P(x) \{ v_1 y_1' + v_2 y_2' \} + Q(x) \{ v_1 y_1 + v_2 y_2 \} = R(x)$$

$$\Rightarrow v_1 \{ y_1'' + P(x) y_1' + Q(x) y_1 \} + v_2 \{ y_2'' + P(x) y_2' + Q(x) y_2 \}$$

$$+ (v_1' y_1' + v_2' y_2') = R(x)$$

$\because$ y_1 and y_2 are linearly independent solution of (2), Therefor.

$$v_1' y_1' + v_2' y_2' = R(x) \quad \text{--- (8)}$$

solving equation (6) & (8), we obtain v_1' and v_2'.

$$\therefore \quad y_1 v_1' + y_2 v_2' + 0 = 0$$

$$y_1' v_1' + y_2' v_2' - R(x) = 0$$

$$\begin{array}{ccccccc} y_2 & & 0 & & y_1 & & y_2 \\ & v_1' & & v_2' & & 1 & \\ y_2' & & -R(x) & & y_1' & & y_2' \end{array}$$

Therefor,

$$\frac{v_1'}{-y_2 R(x)} = \frac{v_2'}{y_1 R(x)} = \frac{1}{y_1 y_2' - y_2 y_1'}$$

$$\therefore \quad v_1' = \frac{-y_2 R(x)}{y_1 y_2' - y_2 y_1'} \quad \text{and} \quad v_2' = \frac{y_1 \cdot R(x)}{y_1 y_2' - y_2 y_1'}$$

$$\therefore \quad v_1 = \int \frac{-y_2 \cdot R(x)}{y_1 y_2' - y_2 y_1'} \, dx \quad \& \quad v_2 = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} \, dx$$

putting the values of v_1 and v_2 in eqn (4)

Which is a particular solution of equation (1)

i,e; $y_p = v_1(x) y_1(x) + v_2(x) \cdot y_2(x)$

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Example (1):- Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \cos x$, by the method of variation of parameters.

Solution:→

Let $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \cos x$ —— (1)

$\therefore$ A.E $\equiv D^2 - 2D = 0$

$\Rightarrow D(D-2) = 0$

$\Rightarrow D = 0, 2$

$\therefore$ C.F. of (1) is $c_1 + c_2 e^{2x}$

(Where c_1 and c_2 are arbitrary constants)

We take $y_1(x) = 1$ and $y_2(x) = e^{2x}$.

And take $y_p = v_1(x) + v_2(x)\, e^{2x}$ —— (2)

$\because$ $y_1(x)$ and $y_2(x)$ are linearly independent, since

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$$

$\therefore y_p' = v_1'(x) + v_2'(x)e^{2x} + 0 + 2v_2(x)\, e^{2x}$

choose $v_1(x)$ and $v_2(x)$ such that

$v_1'(x) + v_2'(x) \cdot e^{2x} = 0$ —— (3)

$\therefore y_p' = 2v_2(x) \cdot e^{2x}$

$\Rightarrow y_p'' = 2v_2'(x)\, e^{2x} + 4v_2(x) \cdot e^{2x}$

putting the values of y_p' and y_p'' in eq^n (1), we have.

5.

$$\because \frac{d^2y_p}{dx^2} - 2\frac{dy_p}{dx} = e^x \cos x$$

$$\Rightarrow 2v_2'(x)\cdot e^{2x} + 4v_2(x)e^{2x} - 2\{2\cdot v_2(x)e^{2x}\} = e^x \cos x$$

$$\Rightarrow 2v_2'(x)\cdot e^{2x} = e^x \cos x$$

$$\Rightarrow v_2'(x) = \frac{e^x \cos x}{2e^{2x}} = \frac{1}{2}e^{-x}\cdot \cos x$$

$$\Rightarrow v_2(x) = \frac{1}{2}\int e^{-x}\cos x\, dx$$

$$= \frac{1}{4}e^{-x}(-\cos x + \sin x) + c_1$$

Now since

$$v_1'(x) = -v_2'(x)\cdot e^{2x} \quad (\text{from } ③)$$

$$= -\frac{1}{2}e^x \cos x$$

$$\therefore v_1(x) = -\frac{1}{4}e^x(\cos x + \sin x) + c_2$$

$\because y_p$ is a solution of ①

$$\therefore y_p = v_1(x) + v_2(x)e^{2x}$$

$$= -\frac{1}{4}e^x(\cos x + \sin x) + c_2$$

$$+ \left\{\frac{1}{4}e^{-x}(-\cos x + \sin x) + c_1\right\}e^{2x}$$

$$\therefore y_p = c_2 + c_1 e^{2x} - \frac{1}{2}e^x \cos x$$

Ans.

Example ② Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + a^2y = \tan ax \quad \text{—— ①}$$

Solution: →

$\because$ A.E $= D^2 + a^2 = 0$

$\Rightarrow D = \pm ia$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$

Let $y_1(x) = \cos ax$ & $y_2(x) = \sin ax$

Let $y_p = v_1(x)\cos ax + v_2(x)\sin ax$ —— (*)

We can choose $v_1(x)$ & $v_2(x)$ in such a way such that

$$\cos ax \, v_1'(x) + \sin ax \, v_2'(x) = 0 \quad \text{—— ②}$$

$$\& \quad -a\sin ax \, v_1'(x) + a\cos ax \, v_2'(x) = \tan ax \quad \text{—— ③}$$

$$\therefore v_1'(x) = -\frac{1}{a}\frac{\sin^2 ax}{\cos ax}$$

$$\& \quad v_2'(x) = \frac{1}{a}\sin ax$$

$$\therefore v_1(x) = -\frac{1}{a}\int(\sec ax - \cos ax)\,dx$$

$$= -\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax + c_1$$

and $v_2(x) = \frac{1}{a}\int \sin ax \, dx$

$$= -\frac{1}{a^2}\cos ax + c_2$$

putting the values of $v_1(x)$ & $v_2(x)$ in (*),
we get

$\because \quad y_p = v_1(x)\cos ax + v_2(x)\sin ax$

$\Rightarrow \quad y_p = c_1\cos ax + c_2\sin ax \div \frac{1}{a^2}\cos ax \log|\sec ax + \tan ax|$

Ans.

Ex 3. Solve by the method of variation of parameters, the equation

$$\frac{d^2y}{dx^2} + y = \sec x.$$

Ex-4. Solve by the method of variation of parameters;

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$$